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NEW PROPERTIES OF THE BUSEMANN ELLIPSE†

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A combination of the Busemann ellipse, the inscribed unit circle and a circle of radius √2 about the same centre is considered. For supersonic two-dimensional potential gas flows, it is shown that the inclinations of the velocity vector in motion along an arbitrary characteristic, the characteristic itself and the characteristic of the other family have values equal to, respectively: the difference between the areas of the elliptical and circular ($R = 1$) sectors, the difference between the areas of the elliptical and circular ($R = \sqrt{2}$) sectors, and the area of the elliptical sector, apart from unimportant multiplicative and additive constants. The straight sides of the sectors in question are the semiminor axis of the ellipse and the radius vector of the velocity. The obvious analogy with one of Kepler's laws is pointed out. The existence of a point of intersection of the ellipse and the second circle illustrates a well-known result of Khristianovich concerning the points of inflexion of characteristics with a monotone velocity distribution. It is shown how the combination of the ellipse and the inscribed circle illustrates the simplification of the compatibility conditions and the Darboux equation for trans- and hypersonic flows. © 1998 Elsevier Science Ltd. All rights reserved.

1. We consider supersonic two-dimensional potential flows of an ideal (non-viscous and non-heat-conducting) gas with adiabatic exponent κ. As is well known [1–3], the Busemann ellipse (Fig. 1) gives a graphical representation of the reduced velocity λ as a function of the Mach angle α. In Fig. 1 the value of λ is represented by the modulus of the radius vector oL and the angle α is measured from the semimajor axis in the clockwise sense. In what follows $\lambda = q/c_*$, where q is the modulus of the velocity vector, c is the velocity of sound, c_* is the critical velocity and $M = q/c = 1/\sin\alpha$ is the Mach number. With this notation, the semimajor and semiminor axes are equal respectively to $od = k = \sqrt{((\varkappa + 1)/(\varkappa - 1))}$, $oc = 1$. Moreover, if an additional vertical line is drawn, parallel to the semimajor axis and passing through the point c (Fig. 1), the graph yields information on the relationship between the reduced velocity, the Mach number and the quantity $m = \sqrt{(M^2 - 1)}$—these three quantities are represented in Fig. 1 by the segments oL, oM and cM.

But the properties of the Busemann ellipse are by no means exhausted by these facts. As it turns out, the graphical combination of the ellipse with two circles of radii 1 and √2 about the centre of the ellipse (Fig. 2) yields information on the dependence of the inclinations of the velocity vector (θ) and the characteristics of the first ($\mu^+ = \theta + \alpha$) and second ($\mu^- = \theta - \alpha$) families on α, λ, M and on one another in motion along an arbitrary characteristic of the first or second family.

We will find it more convenient to use the functions $h(M)$, $f(M)$ and $g(M)$, in terms of which the angles θ, μ^+ and μ^- on the characteristics of the first and second family, respectively, are expressed as follows:

$$\theta = \theta_c + h(M), \quad \mu^+ = \theta_c + \frac{\pi}{2} + g(M), \quad \mu^- = \theta_c - \frac{\pi}{2} + f(M)$$

$$\theta = \theta_c - h(M), \quad \mu^+ = \theta_c + \frac{\pi}{2} - f(M), \quad \mu^- = \theta_c - \frac{\pi}{2} - g(M)$$

where θ_c is the value of θ at the sonic point of the characteristic, and the functions themselves may be written, using the ratio k of the semi-axes of the Busemann ellipse (see above), as follows:

$$h(M) = k \operatorname{arctg}(k^{-1}\sqrt{M^2 - 1}) - \operatorname{arctg}\sqrt{M^2 - 1}$$

$$f(M) = h(M) + \omega, \quad g(M) = h(M) - \omega, \quad \omega = \pi/2 - \alpha$$

Let us consider also the elliptical sector and two circular sectors formed by the semiminor axis oc and its continuation (Fig. 2), the radius vector oL and its continuation and the arcs of the ellipse (sector Loc, of area s), the circle with $R = 1$ (sector L_1oc, of area s_1) and the circle with $R = \sqrt{2}$ (sector L_2oc_2, of area s_2).

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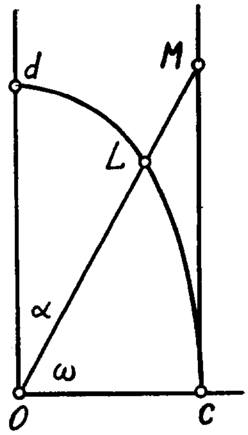

Fig. 1.

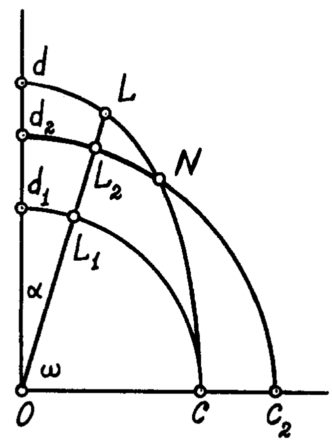

Fig. 2.

Theorem. The functions $h(M)$, $f(M)$ and $g(M)$ may be expressed as follows in terms of the areas of the three sectors

$$h(M)=2(s-s_1),\quad f(M)=2s,\quad g(M)=2(s-s_2)$$

In other words, $h(M)$ is twice the area of the curvilinear triangle LcL_1, $f(M)$ is twice the area of the elliptical sector Loc, and $g(M)$ is twice the difference between the areas of the curvilinear triangles LNL_2 and cNc_2 if $\lambda > \sqrt{2}$ and twice the negative area of the curvilinear quadrilateral LL_2c_2c if $\lambda \leqslant \sqrt{2}$, when oL passes below the point N, including the case where the point d_2 coincides with d or lies below it. The latter occurs when $\varkappa \geqslant 3$.

Proof. Besides the plane of the ellipse, let us consider a plane w passing through the semiminor axis of the ellipse and making an angle φ with the plane of the ellipse such that $\cos\varphi = k^{-1}$. With φ chosen in this way, the projection of the ellipse onto w is a circle of unit radius and the projection of the elliptical sector Loc is a circular sector with angle β and area s_w such that

$$\operatorname{tg}\beta = k^{-1}\operatorname{tg}\omega,\quad s_u=\beta/2=k^{-1}s$$

Hence it follows that $2s = k \operatorname{arctg}(k^{-1}\operatorname{tg}\omega)$. Noting that $\omega = \pi/2 - \alpha = \operatorname{arctg}\sqrt{(M^2-1)}$, $2s_1 = \omega$ and using the formula for $h(M)$, we finally obtain $h(M) = 2(s - s_1)$, which proves the first statement of the theorem. Using the definitions of the functions $f(M)$ and $g(M)$ and the equality $s_2 = 2s_1$, one can readily prove the remaining two statements.

A few corollaries of the theorem follow.

1. Note the analogy with one of Kepler's laws, which is also based on an analysis of the variation in the area of a certain sector cut from an ellipse; except that in Kepler's law the focus of the sector coincides with that of the ellipse [4]. For a deeper analogy with Kepler's law, the first and most significant relationship of the theorem may be reformulated as follows: in motion along an arbitrary characteristic, equal increments of the inclination θ of the velocity vector define equal increments of the areas of the curvilinear quadrilaterals bounded by arcs of the Busemann ellipse, the inscribed circle and the corresponding radius-vectors.

2. The first relationship of the theorem implies a formula for the increments $d\theta = \mp(\lambda^2-1)d\alpha$ along a characteristic, which is clearly equivalent to the more familiar formula $d\theta = \pm\sqrt{(M^2-1)}\, d\ln\lambda$ [1–3, 5, 6]. In what follows, the upper (lower) sign relates to characteristics of the first (second) family.

3. The third relationship of the theorem clearly illustrates and refines a result obtained in [3, 5], according to which, in motion along any characteristic, the derivative of the inclination of the characteristic, $d(\theta \pm \alpha)/d\lambda$, vanishes and changes sign at $\lambda = \sqrt{2}$; we can now add that the last equality is independent of κ, unlike the corresponding equality for $M = 2/\sqrt{(3-\varkappa)}$ [3, 5]. Naturally, this is meaningful only if $\kappa < 3$, when the Busemann ellipse and the circle with $R = \sqrt{2}$ intersect.

2. We will now consider the relation between the geometry of the Busemann ellipse and simplifications of the compatibility conditions and Darboux's equation in both limiting cases, when $\omega = \pi/2 - \alpha \ll 1$ and $\alpha \ll 1$. Indeed, as shown previously, the area s_c of the curvilinear triangle LxL_1 is directly related to the function $h(M)$: $h(M) = 2s_c$. The fact that the arcs of the ellipse and the circles touching at x are second-order curves implies that if $\omega \ll 1$, then $s_c \sim \omega^3$; simple geometric calculations imply that if $\omega \ll 1$, the following forms of the compatibility conditions are equivalent

$$\theta \mp h(M) = \theta \mp 2s_c = \theta \mp \frac{2}{3}\frac{\omega^3}{\varkappa+1} = \theta \mp \frac{2}{3}\frac{(M^2-1)^{3/2}}{\varkappa+1} = \text{const} \tag{2.1}$$

The last of these relationships are known as the compatibility conditions for transonic flows [7].

In the second limiting case, $\alpha \ll 1$, the compatibility conditions are more conveniently expressed in terms of the area s_d of the curvilinear quadrilateral dLL_1d_1 formed by the radius vector, the arcs of the ellipse and the inscribed circle, and the semimajor axis. It is immediately clear that $s_\alpha = \alpha/(\varkappa - 1)$ for $\alpha \ll 1$, whence we obtain the following chain of equivalent compatibility conditions for $\alpha \ll 1$

$$\theta \mp (h(M) - h(\infty)) = \theta \pm 2s_d = \theta \pm \frac{2\alpha}{\varkappa - 1} = \theta \pm \frac{2}{M(\varkappa-1)} = \text{const} \tag{2.2}$$

The last of these relationships are well known in hypersonic flow theory [6].

Let us consider the Darboux equation for the stream function ψ in the plane of the Riemann invariants $\xi = \theta - h(M)$, $\eta = \theta + h(M)$, which holds for supersonic two-dimensional potential flows

$$\psi_{\eta\xi} - G(\eta-\xi)(\psi_\eta - \psi_\xi) = 0, \quad G = \frac{M^4(\varkappa+1)}{8(M^2-1)^{3/2}} \tag{2.3}$$

As it turns out, formulae (2.1) and (2.2) and, in particular, the asymptotic expressions for s_c ($\omega \ll 1$) and s_d ($\alpha \ll 1$) graphically illustrate simplifications of Eq. (2.3) which are obtained when $\omega \ll 1$ and when $\alpha \ll 1$. Indeed, it is obvious from (2.3) that G is inversely proportional to the quantities s_c ($\omega \ll 1$) and s_d ($\alpha \ll 1$). On the other hand, if the Riemann invariants are suitably chosen, s_c and s_d are also directly proportional to their difference.

After simple reductions in the case when $\omega \ll 1$, we obtain expressions for Eq. (2.3) and for the Riemann invariants

$$\begin{gathered}\psi_{\eta\xi} - \frac{1}{24s_c}(\psi_\eta - \psi_\xi) = \psi_{\eta\xi} - \frac{1}{6(\eta-\xi)}(\psi_\eta - \psi_\xi) = 0 \\ \xi = \theta - 2s_c = \theta - \frac{2}{3}\frac{(M^2-1)^{3/2}}{\varkappa+1}, \quad \eta = \theta + 2s_c = \theta + \frac{2}{3}\frac{(M^2-1)^{3/2}}{\varkappa+1}\end{gathered} \tag{2.4}$$

The second of these equations is known as the Euler–Tricomi equation for transonic flows [7].

Now, letting $\alpha \ll 1$, we deduce from (2.2) that a suitable choice of Riemann invariants is provided by ε and δ as given by the formulae

$$\varepsilon = \theta + 2s_d = \theta + \frac{2\alpha}{\varkappa-1}, \quad \delta = \theta - 2s_d = \theta - \frac{2\alpha}{\varkappa-1}$$

after which Eq. (2.3) may be written as

$$\psi_{\delta\varepsilon} - \frac{\varkappa+1}{8(\varkappa-1)s_d}(\psi_\delta - \psi_\varepsilon) = \psi_{\delta\varepsilon} + \frac{\varkappa+1}{2(\varkappa-1)(\delta-\varepsilon)}(\psi_\delta - \psi_\varepsilon) = 0 \tag{2.5}$$

The second of Eqs (2.5) was first derived, by a different method, in [8].

Finally, we write Eqs (2.4) and (2.5) in terms of new variables $\theta, s = s_c$ and $\theta, z = s_d$, respectively

$$\psi_{\theta\theta} - \frac{1}{4}\psi_{ss} - \frac{1}{12s}\psi_s = 0 \quad (\omega \ll 1)$$

$$\psi_{\theta\theta} - \frac{1}{4}\psi_{zz} + \frac{\varkappa+1}{4(\varkappa-1)z}\psi_z = 0 \quad (\alpha \ll 1)$$

A comparison of these equations shows that the singularities obtained as one approaches the sonic line ($s_c = s \to 0$) and the vacuum line ($s_d = z \to 0$) are quite different in nature. This was already observed in [9], using other equations.

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